Fields dynamics and fluxes
Scalar/Vector fields; Gradient; Divergence, LaPlacian; Curl

Here, I gather 5 vectorial calculus concepts frequently used in spacial models.

- **Fields**

A field describes the values of a quantity in space

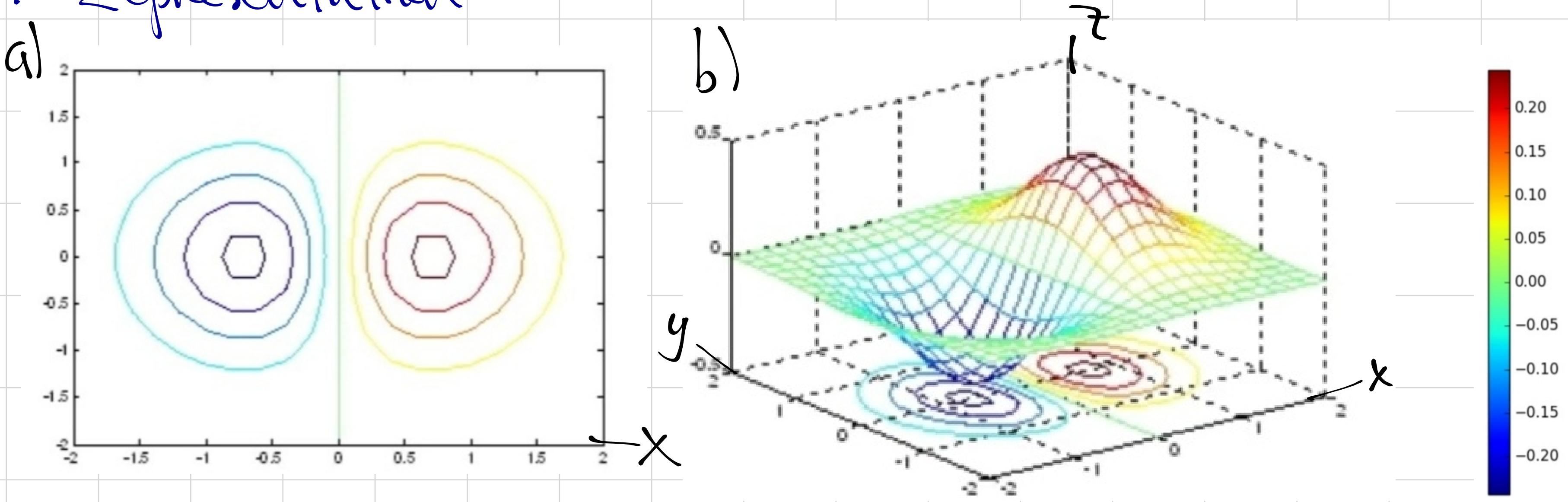
Types (by type of quantity)

- Scalar field
  - When we describe a variable with only magnitude, a scalar, a 1D variable
  - Examples: Temperature, Concentration, etc.
  - The simpliest scalar field is 3D

    $f(x,y) = z$ (1)

    The scalar field can represent the amount of E. Colis (z) in a each point of a biofilm (x, y)
  - But it can have n dependent variables

    $f.(x_1, x_2, \cdots, x_n) = z$
  - Representation



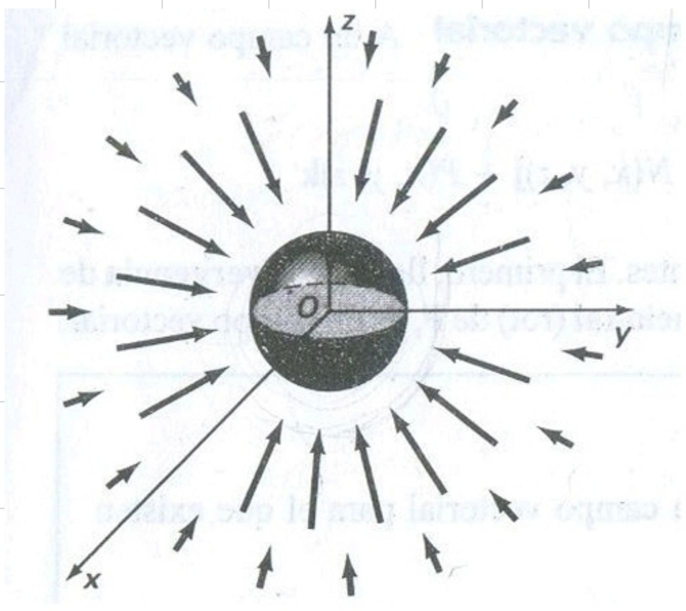
Both a, b represent the same scalar field

In a, z is represented by the colour. In b, z is represented by the colour and height.

- Vector field
  - When the magnitude is a vector
  - Examples: Speed, Gravitational fields
  - Simpliest

    $\vec{F}(x,y) = F_x\hat{\imath} + F_y\hat{\jmath}$

    The equation can represent that, for each point in a biofilm ($F(x,y)$), there is a vetor that describes the speed and direction of the bacterium.
  - Representation



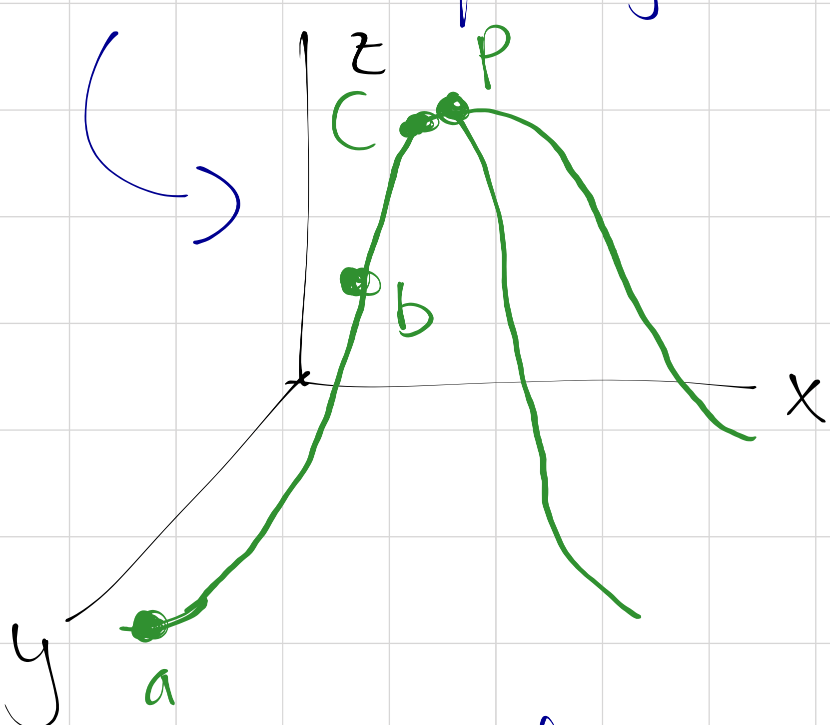
- Gradient of a scalar field (∇)
- Given a scalar field $f(x_1, x_2, \dots, x_n)$, the gradient is a vector field defined as:

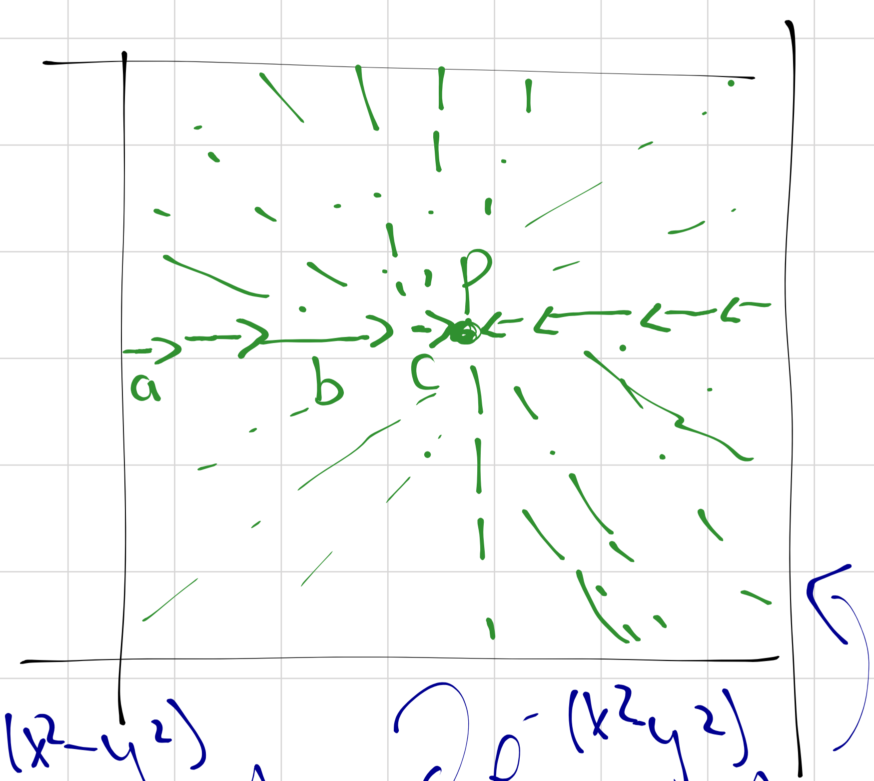
$$\nabla f = \begin{pmatrix} \frac{\partial f}{dx_1} \\ \frac{\partial f}{dx_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix}$$

This vector points to the maximum inclination point

The magnitud of the vector represents the slope at each x, y

- Interpretation
  In a point x, y of a scalar field $f(x,y)=z$,
  The vector gradient points to the zone with higher values (e.g. temperature, concentration)
  The magnitude of the vector represents how much the dependent variable (z) is going to increase if I move towards the direction of the vector.
- Example
  Scalar field: $f(x,y) = e^{-(x^2+y^2)}$





Gradient: $\nabla f(x,y) = \frac{\partial e^{-(x^2-y^2)}}{\partial x}\hat{i}, \frac{\partial e^{-(x^2-y^2)}}{\partial y}\hat{j}$

- **Divergence** ($\nabla \cdot$)
- The divergence of a vector field is a scalar field defined by the inner product ($\cdot$):

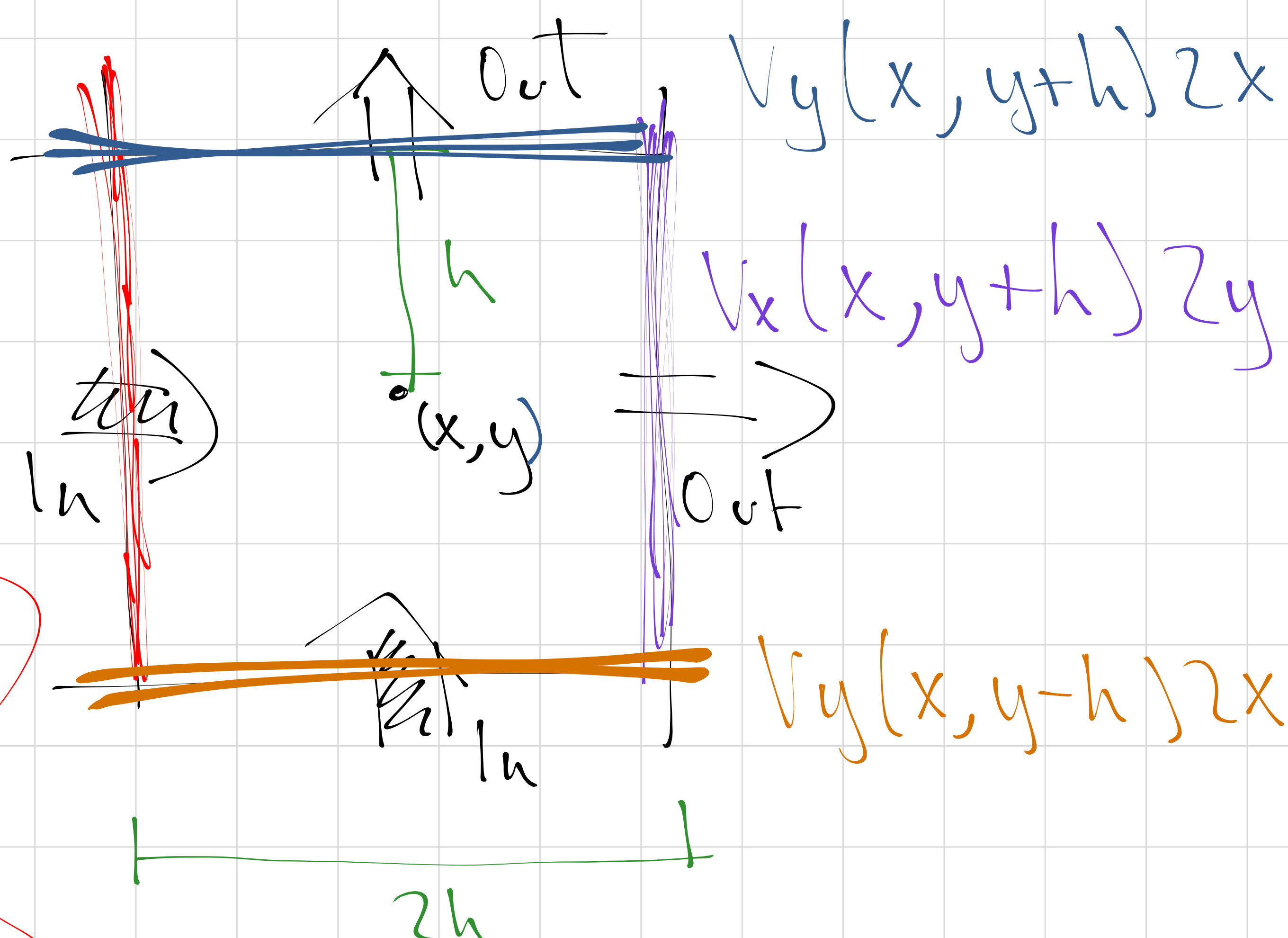
$$\nabla \cdot \vec{V} = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix}^T \begin{pmatrix} V_1 \\ V_2 \\ \vdots \\ \vdots \\ V_{n-1} \\ V_n \end{pmatrix} = \frac{\partial V_1}{\partial x_1} + \frac{\partial V_2}{\partial x_2} + \cdots \frac{\partial V_n}{\partial x_n}$$

- Interpretation:
The divergence of a vector field ($\vec{V}$) in a point (x,y) represents if the adjacent vectors diverge from x, y or converge

$\nabla \cdot \vec{V} > 0$ diverges

$\nabla \cdot \vec{V} < 0$ converges

- Example
We asume that $\vec{V} = V_x, V_y$ represents the flux of bacterium in a bidimensional biofilm in the following way:



The influx is given by:

$$V_x(\underbrace{x-h}_{\text{The particles enter by this arist}}, y)\, 2h \to \text{The total height of the influx door}$$

$\leftrightarrow$ y does not change

Now, we examinate the number of bacterium B in a small area (x, y)
Small because we are going to reduce h
We should consider the influx and outflux

$$\frac{\partial B}{\partial t} = \text{Influx} - \text{Outflux} =$$
$$= 2hV_x(x-h, y) + 2hV_y(x, y-h)$$
$$- 2hV_y(x, y+h) - 2hV_x(x+h, y)$$

$$\text{Concentration}(t) = \frac{B}{(2h)^2} = C$$

$$\frac{\partial C}{\partial t} = \frac{V_x[(x-h, y) - (x+h, y)] + V_y[(x, y-h) - (x, y+h)]}{2h}$$

$h \to 0$

$$\frac{\partial C}{dt} = \lim_{h \to 0}\left(\frac{-V_x(x-h, y) - V_x(x+h, y)}{2h} - \frac{-V_y(x, y-h) - V_y(x, y+h)}{2h}\right) =$$
$$= -\frac{\partial V_x}{\partial x} - \frac{\partial V_y}{\partial y} = -\nabla \cdot V$$

$$\Rightarrow \frac{\partial C}{\partial t} = -\nabla \cdot V$$

if $\nabla \cdot V < 0$ represents the influx
if $\nabla \cdot V > 0$ represents outflux

- **LaPlacian** (Δ or ∇^2)

The LaPlacian of a scalar field f is a scalar field defined by:

$$\nabla^2 f = \nabla \cdot \nabla f$$

The divergence of the gradient of the scalar field

Represents if f is concave or convex

- **Curl** ($\nabla \times$)

- The curl of a vector field, $\vec{V}$ is a set of vectors defined as the scalar product:

$$\vec{V} = V_x \hat{i} + V_y \hat{j} + V_z \hat{j}$$

$$\nabla \times \vec{V} = \left(-\frac{\partial}{\partial z} V_y + \frac{\partial}{\partial y} V_z\right)\hat{i} + \left(\frac{\partial}{\partial z} V_x - \frac{\partial}{\partial x} V_z\right) + \left(-\frac{\partial}{\partial y} V_x + \frac{\partial}{\partial x} V_y\right)$$

- Interpretation

The resulting vector directions defines if the rotation of the vector field. On the other hand, the vector magnitude is correlated with the rotation angle